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Density matrix in quantum electrodynamics, equivalence principle and Hawking effect

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Abstract. The expression for the density matrix describing particles of one sort (electrons or positrons) created by an external electromagnetic field from the vacuum is obtained. The explicit form of the density matrix is found for the case of constant and uniform electric field. Arguments are given for the presence of a connection between the thermal nature of the density matrix describing particles created by the gravitational field of a black hole and the equivalence principle.

1. Introduction

Investigating processes of quantum particle creation by strong gravitational fields of black holes, Hawking (1974, 1975) discovered that the radiation corresponding to these processes had a thermal spectrum. Moreover, it appears that all the correlation functions characterising this radiation coincide with the corresponding functions of thermal radiation, i.e. if we neglect the scattering processes of the created particles in the static gravitational field then the stationary black hole radiation is described by the equilibrium density matrix with the effective temperature $\theta = (8\pi GMk)^{-1}$ where M is the mass of the black hole (see for example Hawking 1976, Parker 1975, Bekenstein and Meisels 1977, Frolov 1977). The origin of the density matrix in this problem can be explained easily. This density matrix is a price which an observer at infinity has to pay for the impossibility registering particles falling into the black hole through the horizon. It is worth noting that the analogous density matrix appears naturally in every case when particles created by an external (not necessary gravitational) field can be divided in two classes ('visible' and 'invisible') and we are interested in finding the average values of observables which depend on the 'visible' particle states only. This density matrix arises as a result of summing over the 'invisible' particle states and as a consequence it does not depend on the particular choice of 'invisible' particle vacuum definition.

On the other hand as far as we know the thermal nature of the density matrix which distinguishes the case of particle creation by the static gravitational field of a black hole still has no simple physical interpretation (see however, Hawking 1976, Gerlach 1976). In this paper we adduce the arguments for the idea that the thermal nature of the black hole radiation may be connected with the equivalence principle. To this end we consider the similar problem in quantum electrodynamics and calculate a density matrix describing particles of one sort (electrons or positrons) created by an electromagnetic field from the vacuum. The solution of this problem is also of interest for investigating processes taking place in strong electromagnetic fields. We will work in the zeroth order in the electron–photon interaction, that is when the electron–positron quantum field interacts with the external electromagnetic field. In this paper we use results and notation adopted in Gitman (1977).

2. Density matrix of particles created by an external field

Let the electron–positron vacuum in an external field at the moment t_1 (t_2) be denoted in the Schrödinger picture as $|0\rangle_{\rm in}$ ($|0\rangle_{\rm out}$) and the corresponding creation and annihilation operators of electrons as $\alpha^\dagger_m, \alpha_m (a^\dagger_m, a_m)$ and positrons as $\beta^\dagger_n, \beta_n (b^\dagger_n, b_n)$. Then the problem of calculating the expectation value at moment t_2 of an observable $f_+ = f_+(a, a^\dagger)$ ($f_- = f_-(b, b^\dagger)$) depending on the states of electrons (positrons) created by an external field from an initial (given at the moment t_1) vacuum state reduces to finding the matrix elements of the form

$$\langle f_\pm\rangle = {}_{\rm in}\langle 0|U^{-1} f_\pm U|0\rangle_{\rm in}, \tag{1}$$

where $U = U(t_2, t_1)$ is the evolution operator for the electron–positron field in the external electromagnetic field. For our purpose it is convenient to transform this expression to the form where the averaging is carried out over out-states. Using the out-states completeness condition at time t_2

$$\sum_{\substack{N,M\\ \{n\}\{m\}}} (N!M!)^{-1} b^\dagger_{n_1} \ldots b^\dagger_{n_N} a^\dagger_{m_1} \ldots a^\dagger_{m_M}|0\rangle_{\rm out\,out}\langle 0|a_{m_M} \ldots a_{m_1} b_{n_N} \ldots b_{n_1} = 1$$

we can obtain the following expansion of the state $U|0\rangle_{\rm in}$ over the out-states:

$$U|0\rangle_{\rm in} = \sum_{\substack{N,M\\ \{n\}\{m\}}} (N!M!)^{-1}{}_{\rm out}\langle\tilde{0}|\tilde{a}_{m_1} \ldots \tilde{a}_{m_M}\tilde{b}_{n_N} \ldots \tilde{b}_{n_1}|0\rangle_{\rm in} b^\dagger_{n_1} \ldots b^\dagger_{n_N} a^\dagger_{m_M} \ldots a^\dagger_{m_1}|0\rangle_{\rm out},$$

where $\tilde{a} = U^{-1}aU$, $\tilde{b} = U^{-1}bU$, $|\tilde{0}\rangle_{\rm out} = U^{-1}|0\rangle_{\rm out}$ are the corresponding quantities in the Heisenberg representation.

As shown in Gitman (1977) the expressions

$${}_{\rm out}\langle\tilde{0}|\tilde{a}_{m_1} \ldots \tilde{a}_{m_M}\tilde{b}_{n_N} \ldots \tilde{b}_{n_1}|0\rangle_{\rm in} \tag{2}$$

can be calculated easily using the techniques of reduction to the generalised normal form. For this purpose all the operators are to be decomposed into two parts one of which annihilates in-vacuum $|0\rangle_{\rm in}$ and the other is responsible for the creation of out-particles from out-vacuum $|\tilde{0}\rangle_{\rm out}$, the creating parts being placed to the left of the annihilating parts. Wick's theorem remains valid if the generalised pairings of the operators A and B are defined as follows

$$\underbracket{AB} = AB - \tilde{N}AB, \qquad \underbracket{AB} = {}_{\rm out}\langle\tilde{0}|AB|0\rangle_{\rm in} C_{\rm v}^{-1},$$

where $\tilde{N}$ is the generalised normal product defined in a conventional way. $C_{\rm v} = {}_{\rm out}\langle 0|U|0\rangle_{\rm in} = {}_{\rm out}\langle\tilde{0}|0\rangle_{\rm in}$ is the probability amplitude for the vacuum to remain a vacuum.

Going back to the definition of the required multi-particle amplitudes (2) we note that $\underbracket{\tilde{a}_m\tilde{a}_{m'}} = \underbracket{\tilde{b}_n\tilde{b}_{n'}} = 0$ and $\underbracket{\tilde{a}_m\tilde{b}_n} = {}_{\rm out}\langle\tilde{0}|\tilde{a}_m\tilde{b}_n|0\rangle_{\rm in} C_{\rm v}^{-1} = w(\overset{+}{m}\overset{-}{n}|0)$ is the relative probability amplitude of electron positron pair creation. Then Wick's theorem obviously implies that ${}_{\text{out}}\langle\tilde{0}|\tilde{a}_{m_1}\ldots\tilde{a}_{m_M}\tilde{b}_{n_N}\ldots\tilde{b}_{n_1}|0\rangle_{\text{in}}$ is the sum of the products of the all possible pairings multiplied by C_{v}. Taking this into account one can obtain

$$U|0\rangle_{\text{in}}=C_{\text{v}}\sum_{N\{n\}\{m\}}(N!)^{-1}\Big(\prod_{i=1}^{N}w(\overset{+}{m_i}\overset{-}{n_i}|0)\Big)b^{\dagger}_{n_1}\ldots b^{\dagger}_{n_N}a^{\dagger}_{m_N}\ldots a^{\dagger}_{m_1}|0\rangle_{\text{out}}.$$

Substituting this expression into (1) and eliminating the operators $b, b^{\dagger}(a, a^{\dagger})$ by moving them through $f_{+}(f_{-})$ to one side we have

$$\langle f_{+}\rangle=P_{\text{v}}\sum_{N\{k\}\{m\}}(N!)^{-1}{}_{\text{out}}\langle 0|a_{k_1}\ldots a_{k_N}f_{+}a^{\dagger}_{m_N}\ldots a^{\dagger}_{m_1}|0\rangle_{\text{out}}\prod_{i=1}^{N}Z^{+}_{m_ik_i},$$

$$\langle f_{-}\rangle=P_{\text{v}}\sum_{N\{l\}\{n\}}(N!)^{-1}{}_{\text{out}}\langle 0|b_{l_N}\ldots b_{l_1}f_{-}b^{\dagger}_{n_1}\ldots b^{\dagger}_{n_N}|0\rangle_{\text{out}}\prod_{i=1}^{N}Z^{-}_{n_il_i},$$

where $P_{\text{v}}=|C_{\text{v}}|^2$ is the probability for the vacuum to remain a vacuum and

$$Z^{+}_{mk}=\sum_{n}w(\overset{+}{m}\overset{-}{n}|0)w^{*}(\overset{+}{k}\overset{-}{n}|0);\qquad Z^{-}_{ne}=\sum_{m}w(\overset{+}{m}\overset{-}{n}|0)w^{*}(\overset{+}{m}\overset{-}{l}|0).$$

Noting that†

$$\exp(W^{+})a^{\dagger}_{k}\exp(-W^{+})=\sum_{m}a^{\dagger}_{m}Z^{+}_{mk},\qquad W^{+}=\sum_{ij}a^{\dagger}_{i}(\ln Z^{+})_{ij}a_{j},$$

$$\exp(W^{-})b^{\dagger}_{l}\exp(-W^{-})=\sum_{n}b^{\dagger}_{n}Z^{-}_{ne},\qquad W^{-}=\sum_{ij}b^{\dagger}_{i}(\ln Z^{-})_{ij}b_{j}.$$

we obtain finally

$$\langle f_{+}\rangle=\sum_{N\{k\}}(N!)^{-1}{}_{\text{out}}\langle 0|a_{k_1}\ldots a_{k_N}f_{+}\rho_{+}a^{\dagger}_{k_N}\ldots a^{\dagger}_{k_1}|0\rangle_{\text{out}}=sp_{+}f_{+}\rho_{+},$$

$$\langle f_{-}\rangle=\sum_{N\{l\}}(N!)^{-1}{}_{\text{out}}\langle 0|b_{l_1}\ldots b_{l_N}f_{-}\rho_{-}b^{\dagger}_{l_N}\ldots b^{\dagger}_{l_1}|0\rangle_{\text{out}}=sp_{-}f_{-}\rho_{-},$$

where

$$\rho_{\pm}=P_{\text{v}}\exp(W^{\pm}) \tag{3}$$

is the density matrix describing the electrons (positrons) created by the field from the vacuum. The expression (3) also remains valid for the case of Klein–Gordon scalar particles created by the external electromagnetic field from the vacuum‡. We note also that the convenient representation of the probability for the vacuum to remain a vacuum follows from the normalisation condition

$$P_{\text{v}}=\det{}^{\eta}(1-\eta Z^{\pm}),\qquad \eta=\begin{cases}1 & \text{for scalar particles}\\ -1 & \text{for spinor particles.}\end{cases}$$

† The known formula

$$\exp(\tau A)B\exp(-\tau A)=\sum_{n=0}^{\infty}\frac{\tau^{n}}{n!}[A\ldots[A,B]\ldots]$$

has been used.

‡ The general expression for the density matrix for scalar particles created by an arbitrary external field can be found in (Frolov 1977).

3. Density matrix in uniform electric field, equivalence principle and Hawking effect

Formula (3) allows one to obtain the explicit form of the density matrices in the cases when the relative probability amplitudes of pair creation are known. For example in the case of the constant uniform electric field E these amplitudes have been found in several papers (Nikishov 1969, Naroshny and Nikishov 1970, Bagrov *et al* 1975, Gitman *et al* 1975). Using the amplitudes obtained one can find

$$Z^{\pm}_{mk} = \delta_{mk}(\exp \pi\lambda + \eta)^{-1}, \tag{4}$$

where the indices m, k represent the momentum $\boldsymbol{p}$ and spin s of spinor particles and $\lambda = \mathscr{E}^2_\perp/|eE|$, $\mathscr{E}_\perp = (m^2 + \boldsymbol{p}^2_\perp)^{1/2}$. Here $\boldsymbol{p}_\perp$ is the component of momentum $\boldsymbol{p}$ perpendicular to the field $\boldsymbol{E}$. Substituting equation (4) into equation (3) we get the explicit form of the density matrix. For instance

$$\rho_+ = P_{\mathrm{v}} \exp\Big(-\sum_m \ln(\exp \pi\lambda + \eta) a^{\dagger}_m a_m\Big). \tag{5}$$

From equation (5) one can calculate the average value of any observable. For example the average number of electrons created by the uniform electric field does not depend on statistics and is of the form

$$w^+_m = sp_+ a^{\dagger}_m a_m \rho_+ = w^-_m = sp_- b^{\dagger}_m b_m \rho_- = \exp(-\pi\lambda). \tag{6}$$

(compare with Bagrov *et al* 1976). The quantity (6) does not depend on the longitudinal (parallel to $\boldsymbol{E}$) component of the momentum. This is due to the fact that since the electric field continues to act on the created particles it becomes equally probable to find particles with any momenta along the field as the time of action grows. Therefore expression (6) contains information only about the motion of the created particles perpendicular to the field. $\mathscr{E}_\perp$ could be taken as the natural measure of the energy of this motion.

Now if one tries to fit the above density matrix to the case of gravitation interaction and replaces formally the electric field strength by the quantity characterising the gravitational field strength κ (surface gravity) and if one uses, the equivalence principle assuming that the coupling constant of the particle with the gravitational field is proportional to the energy of the particle† (i.e. replacing e by $\mathscr{E}_\perp$) then one can obtain formally from (6)

$$w^{\pm}_m = \exp(-\mathscr{E}_\perp \pi/\kappa). \tag{7}$$

The quantity $w^{\pm}_m$ given by (7) could be interpreted as the average number of particles created by the corresponding gravitational field. It should be noted that equation (7) coincides with the Boltzmann distribution with the temperature $\theta' = \kappa/k\pi$. If one takes κ to be the intensity of the gravitational field at the horizon of a black hole then

† The equivalence principle implies that the test particle in the gravitational field moves along a geodesic line in the four-dimensional space–time. When the gravitational field is static, the equation of motion of this test particle can be written in the following three-dimensional form (Landau and Lifshitz 1973):

$$\mathrm{d}p/\mathrm{d}t = \mathscr{E}\kappa$$

where $\kappa = -\mathrm{grad}\ \ln(g_{00})^{1/2}$ is a three-dimensional strength of a gravitational field and

$$\mathscr{E} = mc^2(1 - v^2/c^2)^{1/2}$$

is a total energy of a test particle.

$\theta' = (4\pi GMk)^{-1}$ differs only by the numerical factor from the effective temperature of the black hole. The fact that (7) is the Boltzmann distribution both for Bose and Fermi particles can be explained by taking into account that this distribution describes only perpendicular momentum which does not determine the state of the particle completely. In a similar way the thermal density matrix can be formally obtained from (5) in the limit of high energy ($\lambda \gg 1$) if the principle of equivalence is taken into account.

It should be emphasised that the above consideration can be taken only as an argument in favour of the possible connection between the equivalence principle and the thermal nature of the black hole radiation and of course we do not expect it to be taken as a rigorous proof.

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